

# Classical and new log log-theorems

Alexander Rashkovskii

## Abstract

We present a unified approach to celebrated log log-theorems of Carleman, Wolf, Levinson, Sjöberg, Matsaev on majorants of analytic functions. Moreover, we obtain stronger results by replacing original pointwise bounds with integral ones. The main ingredient is a complete description for radial projections of harmonic measures of strictly star-shaped domains in the plane, which, in particular, explains where the log log-conditions come from.

## 1 Introduction. Statement of results

Our starting point is classical theorems due to Carleman, Wolf, Levinson, and Sjöberg, on majorants of analytic functions.

**Definition 1** *A nonnegative measurable function  $M$  on a segment  $[a, b] \subset \mathbb{R}$  belongs to the class  $\mathcal{L}^{++}[a, b]$  if*

$$\int_a^b \log^+ \log^+ M(t) dt < \infty.$$

(For any real-valued function  $h$ , we write  $h^+ = \max\{h, 0\}$ ,  $h^- = h^+ - h$ .)

Carleman was the first who remarked a special role of functions of the class  $\mathcal{L}^{++}$  in complex analysis, by proving the following variant of the Liouville theorem.

**Theorem A** (T. Carleman [3]) *If an entire function  $f$  in the complex plane  $\mathbb{C}$  has the bound*

$$|f(re^{i\theta})| \leq M(\theta) \quad \forall \theta \in [0, 2\pi], \quad \forall r \geq r_0, \quad (1)$$

*with  $M \in \mathcal{L}^{++}[0, 2\pi]$ , then  $f \equiv \text{const}$ .*

This phenomenon appears also in the Phragmén–Lindelöf setting.

**Theorem B** (F. Wolf [22]) *If a holomorphic function  $f$  in the upper half-plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$  satisfies the condition*

$$\limsup_{z \rightarrow x_0} |f(z)| \leq 1 \quad \forall x_0 \in \mathbb{R}$$

and for any  $\epsilon > 0$  and all  $r > R(\epsilon)$ ,  $\theta \in (0, \pi)$ , one has

$$|f(re^{i\theta})| \leq [M(\theta)]^{\epsilon r}$$

with  $M \in \mathcal{L}^{++}[0, \pi]$ , then  $|f(z)| \leq 1$  on  $\mathbb{C}_+$ .

The most famous statement of this type is the following local result known as the Levinson–Sjöberg theorem.

**Theorem C** (N. Levinson [13], N. Sjöberg [21], F. Wolf [23]) *If a holomorphic function  $f$  in the domain  $Q = \{x + iy : |x| < 1, |y| < 1\}$  has the bound*

$$|f(x + iy)| \leq M(y) \quad \forall x + iy \in Q,$$

*with  $M \in \mathcal{L}^{++}[-1, 1]$ , then for any compact subset  $K$  of  $Q$  there is a constant  $C_K$ , independent of the function  $f$ , such that  $|f(z)| \leq C_K$  in  $K$ .*

For further developments of Theorem C, including higher dimensional variants, see [4], [5], [7], [8], [9]. Theorems A and B were extended to subharmonic functions in higher dimensions in [24].

A similar feature of majorants from the class  $\mathcal{L}^{++}$  was discovered by Beurling in a problem of extension of analytic functions [2]. It also appears in relation to holomorphic functions from the MacLane class in the unit disk [10], [14], and in a description of non-quasi-analytic Carleman classes [6].

The next result, due to Matsaev, does not look like a log log-theorem, however (as will be seen from our considerations) it is also about the class  $\mathcal{L}^{++}$ ; further results in this direction can be found in [16].

**Theorem D** (V.I. Matsaev [15]) *If an entire function  $f$  satisfies the relation*

$$\log |f(re^{i\theta})| \geq -Cr^\alpha |\sin \theta|^{-k} \quad \forall \theta \in (0, \pi), \quad \forall r > 0,$$

*with some  $C > 0$ ,  $\alpha > 1$ , and  $k \geq 0$ , then it has at most normal type with respect to the order  $\alpha$ , that is,  $\log |f(re^{i\theta})| \leq Ar^\alpha + B$ .*

All these theorems can be formulated in terms of subharmonic functions (by taking  $u(z) = \log |f(z)|$  as a pattern), however our main goal is to replace the pointwise bounds like (1) with some integral conditions. A model situation is the following form of the Phragmén–Lindelöf theorem.

**Theorem E** (Ahlfors [1]) *If a subharmonic function  $u$  in  $\mathbb{C}_+$  with nonpositive boundary values on  $\mathbb{R}$  satisfies*

$$\lim_{r \rightarrow \infty} r^{-1} \int_0^\pi u^+(re^{i\theta}) \sin \theta \, d\theta = 0,$$

*then  $u \leq 0$  in  $\mathbb{C}_+$ .*

We will show that all the above theorems are particular cases of results on the class  $\mathcal{A}$  defined below and that the log log-conditions appear as conditions for continuity of certain logarithmic potentials.

**Definition 2** Let  $\nu$  be a probability measure on a segment  $[a, b]$ ; we will identify it occasionally with its distribution function  $\nu(t) = \nu([a, t])$ . Suppose  $\nu(t)$  is strictly increasing and continuous on  $[a, b]$ , and denote by  $\mu$  its inverse function extended to the whole real axis as  $\mu(t) = a$  for  $t < 0$  and  $\mu(t) = b$  for  $t > 1$ . We will say that such a measure  $\nu$  belongs to the class  $\mathcal{A}[a, b]$  if

$$\limsup_{\delta \rightarrow 0} \int_0^\delta \frac{\mu(x+t) - \mu(x-t)}{t} dt = 0. \quad (2)$$

Note that this class is completely different from MacLane's class  $\mathcal{A}$  [14] that consists of holomorphic functions in the unit disk with asymptotic values at a dense subset of the circle. MacLane's class is however described by the condition  $|f(re^{i\theta})| \leq M(r)$ ,  $M \in \mathcal{L}^{++}[0, 1]$ .

Our results extending Theorems A–C and E are as follows.

**Theorem 1** Let a subharmonic function  $u$  in the complex plane satisfy

$$\int_0^{2\pi} u^+(te^{i\theta}) d\nu(\theta) \leq V(t) \quad \forall t \geq t_0, \quad (3)$$

with  $\nu \in \mathcal{A}[0, 2\pi]$  and a nondecreasing function  $V$  on  $\mathbb{R}_+$ . Then there exist constants  $c > 0$  and  $A \geq 1$ , independent of  $u$ , such that

$$u(te^{i\theta}) \leq cV(At) \quad \forall t \geq t_0. \quad (4)$$

**Theorem 2** If a subharmonic function  $u$  in the upper half-plane  $\mathbb{C}_+$  satisfies the conditions

$$\limsup_{z \rightarrow x_0} u(z) \leq 0 \quad \forall x_0 \in \mathbb{R}$$

and

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^\pi u^+(te^{i\theta}) d\nu(\theta) = 0$$

with  $\nu \in \mathcal{A}[0, \pi]$ , then  $u(z) \leq 0 \quad \forall z \in \mathbb{C}_+$ .

**Theorem 3** Let a subharmonic function  $u$  in  $Q = \{x+iy : |x| < 1, |y| < 1\}$  satisfy

$$\int_{-1}^1 u^+(x+iy) d\nu(y) \leq 1 \quad \forall x \in (-1, 1) \quad (5)$$

with  $\nu \in \mathcal{A}[-1, 1]$ . Then for each compact set  $K \subset Q$  there is a constant  $C_K$ , independent of the function  $u$ , such that  $u(z) \leq C_K$  on  $K$ .

Relation of these results to the log log-theorems becomes clear by means of the following statement.

**Definition 3** Denote by  $\mathcal{L}^-[a, b]$  the class of all nonnegative integrable functions  $g$  on the segment  $[a, b]$ , such that

$$\int_a^b \log^- g(s) ds < \infty. \quad (6)$$

**Proposition 1** If the density  $\nu'$  of an absolutely continuous increasing function  $\nu$  belongs to the class  $\mathcal{L}^-[a, b]$ , then  $\nu \in \mathcal{A}[a, b]$ . Consequently, if a holomorphic function  $f$  has a majorant  $M \in \mathcal{L}^{++}$ , then  $\log |f|$  has the corresponding integral bound with the weight  $\nu \in \mathcal{A}$  with the density  $\nu'(t) = \min\{1, 1/M(t)\}$ .

We recall that positive measures  $\nu$  on the unit circle with  $\nu' \in \mathcal{L}^-[0, 2\pi]$  are called *Szegő measures*. Proposition 1 states, in particular, that absolutely continuous Szegő measures belong to the class  $\mathcal{A}[0, 2\pi]$ .

An integral version of Theorem D has the following form.

**Theorem 4** Let a function  $u$ , subharmonic in  $\mathbb{C}$  and harmonic in  $\mathbb{C} \setminus \mathbb{R}$ , satisfy the inequality

$$\int_{-\pi}^{\pi} u^-(re^{i\theta}) \Phi(|\sin \theta|) d\theta \leq V(r) \quad \forall r \geq r_0, \quad (7)$$

where  $\Phi \in \mathcal{L}^-[0, 1]$  is nondecreasing and the function  $V$  is such that  $r^{-1-\delta}V(r)$  is increasing in  $r$  for some  $\delta > 0$ . Then there are constants  $c > 0$  and  $A \geq 1$ , independent of  $u$ , such that

$$u(re^{i\theta}) \leq cV(Ar) \quad \forall r \geq r_1 = r_1(u).$$

Our proofs of Theorems 1–4 rest on a presentation of measures of the class  $\mathcal{A}[0, 2\pi]$  as radial projections of harmonic measures of star-shaped domains. Let  $\Omega$  be a bounded Jordan domain containing the origin. Given a set  $E \subset \partial\Omega$ ,  $\omega(z, E, \Omega)$  will denote the harmonic measure of  $E$  at  $z \in \Omega$ , i.e., the solution of the Dirichlet problem in  $\Omega$  with the boundary data 1 on  $E$  and 0 on  $\partial\Omega \setminus E$ . The measure  $\omega(0, E, \Omega)$  generates a measure on the unit circle  $\mathbb{T}$  by means of the radial projection  $\zeta \mapsto \zeta/|\zeta|$ . It is convenient for us to consider it as a measure on the segment  $[0, 2\pi]$ , so we put

$$\hat{\omega}_\Omega(F) = \omega(0, \{\zeta \in \partial\Omega : \arg \zeta \in F\}, \Omega) \quad (8)$$

for each Borel set  $F \subset [0, 2\pi]$ .

The inverse problem is as follows. Given a probability measure on the unit circle  $\mathbb{T}$ , is it the radial projection of the harmonic measure of any domain  $\Omega$ ?

For our purposes we specify  $\Omega$  to be *strictly star-shaped*, i.e., of the form

$$\Omega = \{re^{i\theta} : r < r_\Omega(\theta), 0 \leq \theta \leq 2\pi\} \quad (9)$$

with  $r_\Omega$  a positive continuous function on  $[0, 2\pi]$ ,  $r_\Omega(0) = r_\Omega(2\pi)$ .

**Theorem 5** *A continuous probability measure  $\nu$  on  $[0, 2\pi]$  is the radial projection of the harmonic measure of a strictly star-shaped domain if and only if  $\nu \in \mathcal{A}[0, 2\pi]$ .*

**Corollary 4** *Every absolutely continuous measure from the Szegő class on the unit circle is the radial projection of the harmonic measure of some strictly star-shaped domain.*

Theorem 5 is proved by a method originated by B.Ya. Levin in theory of majorants in classes of subharmonic functions [11].

Theorems 1–3 and 5 (some of them in a slightly weaker form) were announced in [18] and proved in [19] and [20]. The main objective of the present paper, Theorem 4, is new. Since its proof rests heavily on Theorem 5, we present a proof of the latter as well, having in mind that the papers [19] and [20] are not easily accessible. Moreover, we include the proofs of Theorems 1–3, too, motivated by the same accessibility reason as well as by the idea of showing the whole picture.

## 2 Radial projections of harmonic measures (Proofs of Theorem 5 and Proposition 1)

Measures from the class  $\mathcal{A}$  have a simple characterization as follows.

**Proposition 2** *Let  $\mu$  and  $\nu$  be as in Definition 2. Then the function*

$$N(x) = \int_0^1 \log |x - t| d\mu(t)$$

*is continuous on  $[0, 1]$  if and only if  $\nu \in \mathcal{A}[a, b]$ .*

*Proof.* The function  $N(x)$  is continuous on  $[0, 1]$  if and only if for any  $\epsilon > 0$  one can choose  $\delta \in (0, 1)$  such that

$$I_x(\delta) = \int_{|t-x|<\delta} \log |x - t| d\mu(t) > -\epsilon$$

for all  $x \in [0, 1]$ . Integrating  $I_x$  by parts, we get

$$|I_x(\delta)| = \int_0^\delta \frac{r_x(t)}{t} dt + r_x(\delta) |\log \delta|,$$

where  $r_x(t) = \mu(x+t) - \mu(x-t)$ . Therefore, continuity of  $N(x)$  implies (2). On the other hand, since  $r_x(t)$  increases in  $t$ , we have

$$r_x(\delta) |\log \delta| = 2r_x(\delta) \int_\delta^{\sqrt{\delta}} \frac{dt}{t} \leq 2 \int_\delta^{\sqrt{\delta}} \frac{r_x(t)}{t} dt,$$

which gives the reverse implication. □

In the proof of Theorem 5, we will use this property in the following form.

**Proposition 3** *Let  $\mu$  and  $\nu$  be as in Definition 2 for the class  $\mathcal{A}[0, 2\pi]$ . Then the function*

$$h(z) = \int_0^{2\pi} \log |e^{i\theta} - z| d\mu(\theta/2\pi)$$

*is continuous on  $\mathbb{T}$  if and only if  $\nu \in \mathcal{A}[0, 2\pi]$ .*

*Proof of Theorem 5.* 1) First we prove the sufficiency: every  $\nu \in \mathcal{A}[0, 2\pi]$  has the form  $\nu = \widehat{\omega}_\Omega$  (8) for some strictly star-shaped domain  $\Omega$ . In particular, for any compact set  $K \in \Omega$  there is a constant  $C(K)$  such that

$$\omega(z, E, \Omega) \leq C(K) \nu(\arg E) \quad \forall z \in E \quad (10)$$

for every Borel set  $E \subset \partial\Omega$ , where  $\arg E = \{\arg \zeta : \zeta \in E\}$ .

Let

$$u(z) = \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\theta} - z| d\mu(\theta/2\pi)$$

with  $\mu$  the inverse function to  $\nu \in \mathcal{A}[0, 2\pi]$ . The function  $u$  is subharmonic in  $\mathbb{C}$  and harmonic outside the unit circle  $\mathbb{T}$ . By Proposition 3, it is continuous on  $\mathbb{T}$  and thus, by Evans' theorem, in the whole plane. Let  $v$  be a harmonic conjugate to  $u$  in the unit disk  $\mathbb{D}$ , which is determined uniquely up to a constant. Since  $u \in C(\overline{\mathbb{D}})$ , radial limits  $v^*(e^{i\psi})$  of  $v$  exist a.e. on  $\mathbb{T}$ . Let us fix such a point  $e^{i\psi_0}$  and choose the constant in the definition of  $v$  in such a way that  $v^*(e^{i\psi_0}) = \psi_0$ .

Consider then the function  $w(z) = z \exp\{-u(z) - iv(z)\}$ ,  $z \in \mathbb{D}$ . By the Cauchy-Riemann condition,  $\partial v / \partial \phi = r \partial u / \partial r$ , which implies

$$\begin{aligned} \arg w(re^{i\psi}) &= \psi - v(re^{i\psi_0}) - \int_{\psi_0}^{\psi} \frac{\partial v(re^{i\phi})}{\partial \phi} d\phi = \psi_0 - v(re^{i\psi_0}) \\ &+ \frac{1}{2\pi} \int_{\psi_0}^{\psi} \int_0^{2\pi} \left[ 1 - \frac{2r^2 - 2r \cos(\theta - \phi)}{|r - e^{i(\theta - \phi)}|^2} \right] d\mu(\theta/2\pi) d\phi \\ &= \psi_0 - v(re^{i\psi_0}) + \frac{1}{2\pi} \int_{\psi_0}^{\psi} \int_0^{2\pi} \frac{1 - r^2}{|r - e^{i(\theta - \phi)}|^2} d\mu(\theta/2\pi) d\phi. \end{aligned}$$

By changing the integration order and passing to the limit as  $r \rightarrow 1$ , we derive that for each  $\psi \in [0, 2\pi]$  there exists the limit

$$\lim_{r \rightarrow 1} \arg w(re^{i\psi}) = \mu(\psi/2\pi) - \mu(\psi_0/2\pi).$$

Therefore the function  $\arg w$  is continuous up to the boundary of the disk; in particular, we can take  $\psi_0 = 0$ . Since  $|w|$  is continuous in  $\overline{\mathbb{D}}$  as well, so is  $w$ .

By the boundary correspondence principle,  $w$  gives a conformal map of  $\mathbb{D}$  onto the domain

$$\Omega = \{re^{i\theta} : r < \exp\{-u(\exp\{2\pi i\nu(\theta)\})\}, 0 \leq \theta \leq 2\pi\}. \quad (11)$$

It is easy to see that the domain  $\Omega$  is what we sought. Let  $f$  be the conformal map of  $\Omega$  to  $\mathbb{D}$ , inverse to  $w$ . For  $z \in \Omega$  and  $E \subset \partial\Omega$ , we have

$$\begin{aligned}\omega(z, E, \Omega) &= \omega(f(z), f(E), U) = \frac{1}{2\pi} \int_{\arg f(E)} \frac{1 - |f(z)|^2}{|f(z) - e^{it}|^2} dt \\ &= (1 - |f(z)|^2) \int_{\arg E} \frac{d\nu(s)}{|f(z) - e^{2\pi i\nu(s)}|^2},\end{aligned}$$

which proves the claim.

2) Now we prove the necessity: if  $\omega$  is of the form (9), then  $\widehat{\omega}_\Omega \in \mathcal{A}[0, 2\pi]$ .

We use an idea from the proof of [11, Theorem 2.4]. Let  $w$  be a conformal map of  $\mathbb{D}$  to  $\Omega$ ,  $w(0) = 0$ . Since  $\Omega$  is a Jordan domain,  $w$  extends to a continuous map from  $\overline{\mathbb{D}}$  to  $\overline{\Omega}$ , and we can specify it to have  $\arg w(1) = 0$ . Define

$$f(z) = u(z) + iv(z) = \log \frac{w(z)}{z} \text{ for } |z| \leq 1, \quad f(z) = f(|z|^{-2}z) \text{ for } |z| > 1.$$

It is analytic in  $\mathbb{D}$  and continuous in  $\mathbb{C}$ . Define then the function

$$\lambda(z) = u(z) + \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| dv(e^{i\psi}), \quad (12)$$

$\delta$ -subharmonic in  $\mathbb{C}$  and harmonic in  $\mathbb{C} \setminus \mathbb{T}$ . Let us show that it is actually harmonic (and, hence, continuous) everywhere. To this end, take any function  $\alpha \in C(\mathbb{T})$  and a number  $r < 1$ , and apply Green's formula for  $u(z)$  and  $A(z) = |z|\alpha(z/|z|)$  in the domain  $D_r = \{r < |z| < r^{-1}\}$ :

$$\int_{D_r} (A\Delta u - u\Delta A) = \left[ \frac{\rho}{2\pi} \int_0^{2\pi} \left( \rho\alpha(e^{i\psi}) \frac{\partial u(\rho e^{i\psi})}{\partial \rho} - u(\rho e^{i\psi})\alpha(e^{i\psi}) \right) d\psi \right]_{\rho=r}^{\rho=R}. \quad (13)$$

Using the definition of the function  $f$  outside  $\mathbb{D}$  and the Cauchy-Riemann equations  $\partial v/\partial \phi = \rho \partial u/\partial \rho$  if  $\rho < 1$  and  $\partial v/\partial \phi = -\rho \partial u/\partial \rho$  if  $\rho > 1$  (which follows from the definition of  $f$ ), we can write the right hand side of (13) as

$$-\frac{r + r^{-1}}{2\pi} \int_0^{2\pi} \alpha(e^{i\psi}) d_\psi v(re^{i\psi}) + \frac{r - r^{-1}}{2\pi} \int_0^{2\pi} u(re^{i\psi}) \alpha(e^{i\psi}) d\psi.$$

When  $r \rightarrow 1$ , (13) takes the form

$$\int_{\mathbb{T}} \alpha \Delta u = -\frac{1}{\pi} \int_0^{2\pi} \alpha(e^{i\psi}) dv(e^{i\psi}),$$

which implies the harmonicity of the function  $\lambda(z)$  (12) in the whole plane.

Now we recall that  $v(e^{i\psi}) = \arg w(e^{i\psi}) - \psi$ . Since the harmonic measure of the  $w$ -image of the arc  $\{e^{i\theta} : 0 < \theta < \psi\}$  equals  $\psi/2\pi$ , we have

$$\widehat{\omega}_\Omega(\arg w(e^{i\psi})) = \psi/2\pi$$

and thus  $\arg w(e^{i\psi}) = \mu(\psi/2\pi)$  with  $\mu$  the inverse function to  $\widehat{\omega}_\Omega(\psi)$ . Therefore,  $v(e^{i\psi}) = \mu(\psi/2\pi) - \psi$ .

Consider, finally, the function

$$\gamma(z) = \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| d\mu(\psi/2\pi) = \lambda(z) - u(z) + \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| d\psi.$$

Since it is continuous on  $\mathbb{T}$ , Proposition 3 implies  $\widehat{\omega}_\Omega \in \mathcal{A}[0, 2\pi]$ , and the theorem is proved.  $\square$

Note that all the dilations  $t\Omega$  of  $\Omega$  ( $t > 0$ ) represent the same measure from  $\mathcal{A}[0, 2\pi]$ , and  $\Omega$  with a given projection  $\widehat{\omega}_\Omega$  is unique up to the dilations.

Now we prove Proposition 1 that presents a wide subclass of  $\mathcal{A}$  with a more explicit description.

*Proof of Proposition 1.* Let  $\nu : [0, 1] \rightarrow [0, 1]$  be an absolutely continuous, strictly increasing function,  $\nu' \in \mathcal{L}^-[0, 1]$ . Since  $\text{mes}\{t : \nu'(t) = 0\} = 0$ , its inverse function  $\mu$  is absolutely continuous ([17], p. 297), so

$$\mu(t) = \int_0^t g(s) ds$$

with  $g$  a nonnegative function on  $[0, 1]$ . We have

$$\infty > \int_0^1 \log^- \nu'(t) dt = \int_0^1 \log^- \frac{1}{\mu'(t)} d\mu(t) = \int_0^1 g(t) \log^+ g(t) dt,$$

so  $g$  belongs to the Zygmund class  $\mathbf{L} \log \mathbf{L}$ .

Let  $\Delta(t)$  denote the modulus of continuity of the function  $\mu$ . Note that it can be expressed in the form

$$\Delta(t) = \int_0^t h(s) ds$$

where  $h$  is the nonincreasing equimeasurable rearrangement of  $g$ . Then

$$\begin{aligned} \int_0^1 \frac{\Delta(t)}{t} dt &= \int_0^1 t^{-1} \int_0^1 h(s) ds dt = \int_0^1 h(s) \log s^{-1} ds \\ &= \int_{E_1 \cup E_2} h(s) \log s^{-1} ds, \end{aligned}$$

where  $E_1 = \{s \in (0, 1) : h(s) > s^{-1/2}\}$ ,  $E_2 = (0, 1) \setminus E_1$ . Since  $h \in \mathbf{L} \log \mathbf{L}[0, 1]$ ,

$$\int_{E_1} h(s) \log s^{-1} ds \leq 2 \int_{E_1} h(s) \log h(s) ds < \infty.$$



Besides,

$$\int_{E_2} h(s) \log s^{-1} ds \leq \int_{E_2} s^{-1/2} \log s^{-1} ds < \infty.$$

Therefore,

$$\int_0^1 \frac{\Delta(t)}{t} dt < \infty$$

and thus

$$\lim_{\delta \rightarrow 0} \int_0^\delta \frac{\Delta(t)}{t} dt = 0,$$

which gives (2).  $\square$

Corollary 4 follows directly from the definition of the Szegő class, Theorem 5 and Proposition 1.

### 3 Proofs of Theorems 1 and 2

Here we show how the integral variants of Carleman's and Wolf's theorems can be derived from Theorem 5.

We will need an elementary

**Lemma 5** *Let  $r(\theta) \in C[0, 2\pi]$ ,  $1 < r_1 \leq r(\theta) \leq r_2$ , let  $\nu$  be a positive measure on  $[0, 2\pi]$  and  $V(t)$  be a nonnegative function on  $[0, \infty]$ . If a nonnegative function  $v(te^{i\theta})$  satisfies*

$$\int_0^{2\pi} v(te^{i\theta}) d\nu(\theta) \leq V(t) \quad \forall t \geq t_0,$$

*then for any  $R_2 > R_1 \geq t_0$ ,*

$$\int_{R_1}^{R_2} \int_0^{2\pi} v(tr(\theta)e^{i\theta}) d\nu(\theta) dt \leq r_1^{-1} \int_{r_1 R_1}^{r_2 R_2} V(t) dt.$$

*Proof of Lemma 5* is straightforward:

$$\begin{aligned} \int_{R_1}^{R_2} \int_0^{2\pi} v(tr(\theta)e^{i\theta}) d\nu(\theta) dt &= \int_0^{2\pi} \int_{R_1 r(\theta)}^{R_2 r(\theta)} v(te^{i\theta}) dt \frac{d\nu(\theta)}{r(\theta)} \\ &\leq r_1^{-1} \int_0^{2\pi} \int_{R_1 r_1}^{R_2 r_2} v(te^{i\theta}) dt d\nu(\theta) \leq r_1^{-1} \int_{R_1 r_1}^{R_2 r_2} V(t) dt. \end{aligned}$$

$\square$

*Proof of Theorem 1.* By Theorem 5, there exists a domain  $\Omega$  of the form (9) that contains  $\overline{\mathbb{D}}$  such that

$$\omega(z, E, \Omega) \leq c_1 \nu(\arg E), \quad \forall z \in \overline{\mathbb{D}}, \quad E \subset \partial\Omega, \quad (14)$$

with a constant  $c_1 > 0$ , see (10). Let  $r_1 = \min r(\theta)$ .  $r_2 = \max r(\theta)$ . By the Poisson–Jensen formula applied to the function  $v_t(z) = u^+(tz)$  ( $t > 0$ ) in the domain  $s\Omega$  ( $s > 1$ ) we have, due to (14),

$$\begin{aligned} v_t(z) &\leq \int_{\partial s\Omega} v_t(\zeta) \omega(z, d\zeta, s\Omega) = \int_{\partial\Omega} v_t(s\zeta) \omega(s^{-1}z, d\zeta, \Omega) \\ &\leq c_1 \int_0^{2\pi} v_t(s r(\theta) e^{i\theta}) d\nu(\theta), \quad z \in \overline{\mathbb{D}}. \end{aligned}$$

The integration of this relation over  $s \in [1, R]$  ( $R > 1$ ) gives, by Lemma 5,

$$(R-1)v_t(z) \leq c_1 \int_1^R \int_0^{2\pi} v_t(s r(\theta) e^{i\theta}) d\nu(\theta) ds \leq c_2 t^{-1} r_1^{-1} \int_{tr_1}^{tr_2 R} V(s) ds$$

for each  $t \geq t_0$ . So,

$$u(te^{i\theta}) \leq c(R)V(tr_2 R), \quad t \geq t_0,$$

which proves the theorem.  $\square$

**Remarks.** 1. It is easy to see that the constant  $A$  in (4) can be chosen arbitrarily close to  $r_2/r_1 \geq 1$ .

2. Note that we have used inequality (3) in the integrated form only, so the following statement is actually true: *If a subharmonic function  $u$  on  $\mathbb{C}$  satisfies*

$$\int_{t_0}^t \int_0^{2\pi} u^+(se^{i\theta}) d\nu(\theta) ds \leq W(t) \quad \forall t \geq t_0 \quad (15)$$

*with  $\nu \in \mathcal{A}[0, 2\pi]$  and a nondecreasing function  $W$ , then there are constants  $c > 0$  and  $A \geq 1$ , independent of  $u$ , such that  $u(te^{i\theta}) \leq ct^{-1}W(At)$  for all  $t \geq t_0$ .*

Now we prove Theorem 2 as a consequence of Theorem 1.

*Proof of Theorem 2.* The function  $v$  equal to  $u^+$  in  $\mathbb{C}_+$  and 0 in  $\mathbb{C} \setminus \mathbb{C}_+$  is a subharmonic function in  $\mathbb{C}$  satisfying the condition

$$\int_0^{2\pi} v^+(te^{i\theta}) d\nu(\theta) \leq V_1(t)$$

with  $\nu \in \mathcal{A}[0, 2\pi]$  and  $V_1(t) = o(t)$ ,  $t \rightarrow \infty$ . Therefore, it satisfies the conditions of Theorem 1 with the majorant  $V(t) = \sup\{V_1(s) : s \leq t\}$ . So,  $\sup_{\theta} u^+(te^{i\theta}) = o(t)$  as  $t \rightarrow \infty$ , and the conclusion holds by the standard Phragmén–Lindelöf theorem.  $\square$

## 4 Proof of Theorem 3

The integral version of the Levinson–Sjöberg theorem will be proved along the same lines as Theorem 1, however the local situation needs a more refined adaptation.

We start with two elementary statements close to Lemma 5.

**Lemma 6** *Let a nonnegative integrable function  $v$  in the square  $Q = \{|x|, |y| < 1\}$  satisfy (5) with a continuous strictly increasing function  $\nu$ . Then for any  $d \in (0, 1)$  there exists a constant  $M_1(d)$ , independent of  $u$ , such that for each  $y_0 \in (-1, 1)$  one can find a point  $y_1 \in (-1, 1) \cap (y_0 - d, y_0 + d)$  with*

$$\int_{-1}^1 v(x + iy_1) dx < M_1(d).$$

*Proof.* Assume  $y_0 \geq 0$ , then

$$\int_{y_0-d}^{y_0} \int_{-1}^1 v(x + iy) dx d\nu(y) = \int_{-1}^1 \int_{y_0-d}^{y_0} v(x + iy) d\nu(y) dx \leq 2.$$

Therefore for some  $y_1 \in (y_0 - d, y_0)$ ,

$$\int_{-1}^1 v(x + iy_1) dx \leq 2[\nu(y_0) - \nu(y_0 - d)]^{-1} \leq 2[\Delta_*(\nu, d)]^{-1}$$

with  $\Delta_*(\nu, d) = \inf\{\nu(t) - \nu(t - d) : t \in (0, 1)\} > 0$ . □

**Lemma 7** *Let a function  $v$  satisfy the conditions of Lemma 6, a function  $r$  be continuous on a segment  $[a, b] \subset [-1, 1]$ ,  $0 < r_1 = \min r(y) \leq \max r(y) = r_2 < 1$ , and  $\delta \in (0, 1 - r_2)$ . Then there exists  $t \in (0, \delta)$  such that*

$$\int_a^b v(t + r(y) + iy) d\nu(y) < M_2(\delta)$$

with  $M_2(\delta)$  independent of  $v$ .

*Proof.* We have

$$\begin{aligned} \int_0^\delta \int_a^b v(t + r(y) + iy) d\nu(y) &= \int_a^b \int_{r(y)}^{\delta+r(y)} v(s + iy) ds d\nu(y) \\ &\leq \int_{r_1}^{\delta+r_2} \int_a^b v(s + iy) d\nu(y) ds \leq \delta + r_2 - r_1. \end{aligned}$$

Thus one can find some  $t \in (0, \delta)$  such that

$$\int_a^b v(t + r(y) + iy) d\nu(y) < \delta^{-1}(\delta + r_2 - r_1).$$

□

*Proof of Theorem 3.* Consider the measure  $\nu_1$  on  $[-i, i]$  defined as

$$\nu_1(E) = \nu(-iE), \quad E \subset [-i, i].$$

The conformal map  $f(z) = \exp\{z\pi/2\}$  of the strip  $\{|\operatorname{Im} z| < 1\}$  to the right half-plane  $\mathbb{C}_r$  pushes the measure  $\nu_1$  forward to the measure  $f^*\nu$  on the semicircle  $\{e^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\}$ , producing a measure of the class  $\mathcal{A}[-\pi/2, \pi/2]$ ; we extend it to some measure  $\nu_2 \in \mathcal{A}[-\pi, \pi]$ . By Theorem 5, there is a strictly star-shaped domain  $\Omega \supset \overline{\mathbb{D}}$  such that the radial projection of its harmonic measure at 0 is the normalization  $\nu_2/\nu_2([-\pi, \pi])$  of  $\nu_2$ .

Let  $\Omega_1 = \Omega \cap \mathbb{C}_r$ , then for every Borel set  $E \subset \Gamma = \partial\Omega_1 \cap \mathbb{C}_r$  and any compact set  $K \subset \Omega_1$ ,

$$\omega(w, E, \Omega_1) \leq C_1(K) \nu_2(\arg E) \quad \forall w \in K.$$

The pre-image  $\Omega_2 = f^{[-1]}(\Omega_1)$  of  $\Omega_1$  has the form

$$\Omega_2 = \{z = x + iy : x < \varphi(y), y \in (0, 1)\}$$

with some function  $\varphi \in C[-1, 1]$ . Let

$$\Gamma_2 = \{x + iy : x = \varphi(y), y \in (0, 1)\},$$

then for every Borel  $E \subset \Gamma_2$  and any compact subset  $K$  of  $\Omega_2$ ,

$$\omega(z, E, \Omega_2) \leq C_2(K) \nu(\operatorname{Im} E) \quad \forall z \in K. \quad (16)$$

For the domain

$$\Omega_3 = \{z = x + iy : x > -\varphi(y), y \in (0, 1)\}$$

we have, similarly, the relation

$$\omega(z, E, \Omega_3) \leq C_3(K) \nu(\operatorname{Im} E) \quad \forall z \in K \quad (17)$$

for each  $E \subset \Gamma_3 = \{x + iy : x = -\varphi(y), y \in (0, 1)\}$  and compact set  $K \subset \Omega_3$ .

Let now  $K$  be an arbitrary compact subset of the square  $Q$ . We would be almost done if we were able to find some reals  $h_2(K)$  and  $h_3(K)$  such that

$$K \subset \{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\} \subset \overline{\{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\}} \subset Q.$$

However this is not the case for any  $K$  unless  $\varphi \equiv \text{const}$ . That is why we need partition.

Given  $K$  compactly supported in  $Q$ , choose a positive  $\lambda < (4 \operatorname{dist}(K, \partial Q))^{-1}$  and then  $\tau \in (0, \lambda)$  such that the modulus of continuity of  $\varphi$  at  $4\tau$  is less than  $\lambda$ . Take a finite covering of  $K$  by disks  $B_j = \{z : |z - z_j| < \tau\}$ ,  $z_j \in K$ ,  $1 \leq j \leq n$ . To prove the theorem, it suffices to estimate the function  $u$  on each  $B_j$ .

Let  $Q_j = \{z \in Q : |\operatorname{Im}(z - z_j)| < 2\tau\}$ , then  $B_j \subset Q_j$  and  $\operatorname{dist}(B_j, \partial Q_j) = \tau$ . Take also

$$\Omega_2^{(j)} = \Omega_2 \cap Q_j, \quad \Gamma_2^{(j)} = \Gamma_2 \cap \overline{\Omega_2^{(j)}} = \{x + iy : x = \varphi(y), a_j \leq y \leq b_j\}.$$

Now we can find reals  $h_2^{(j)}$  and  $h_3^{(j)}$  such that

$$\Gamma_2^{(j)} + h_2^{(j)} = \{x + iy : x = r_2^{(j)}(y)\} \subset Q_j \cap \{x + iy : 1 - 4\lambda < x < 1 < 2\lambda\}$$

and

$$\Gamma_3^{(j)} + h_3^{(j)} = \{x + iy : x = r_3^{(j)}(y)\} \subset Q_j \cap \{x + iy : -1 + 2\lambda < x < -1 + 4\lambda\}.$$

Furthermore, by Lemma 7, there exist  $t_2^{(j)} \in (0, \lambda)$  and  $t_3^{(j)} \in (-\lambda, 0)$  such that

$$\int_{a_j}^{b_j} u^+(t_k^{(j)} + r_k^{(j)}(y) + iy) d\nu(y) < M_2(\lambda), \quad k = 2, 3. \quad (18)$$

Finally we can find, due to Lemma 6,  $y_1^{(j)} \in (a_j, a_j + \tau)$  and  $y_2^{(j)} \in (b_j - \tau, b_j)$  such that

$$\int_{-1}^1 u^+(x + iy_m) dx < M_1(\tau), \quad m = 1, 2. \quad (19)$$

Denote

$$\Omega^{(j)} = \{x + iy : r_3^{(j)}(y) + t_3^{(j)} < x < r_2^{(j)}(y) + t_2^{(j)}, y_1^{(j)} \leq y \leq y_2^{(j)}\}.$$

Since  $\overline{B_j} \subset \Omega^{(j)}$ , relations (16) and (17) imply

$$\omega(z, E, \Omega^{(j)}) \leq C(B_j) \nu(\text{Im } E) \quad \forall z \in B_j \quad (20)$$

for all  $E$  in the vertical parts of  $\partial\Omega^{(j)}$ . For  $E$  in the horizontal parts of  $\partial\Omega^{(j)}$ , we have, evidently,

$$\omega(z, E, \Omega^{(j)}) \leq C(B_j) \text{mes } E \quad \forall z \in B_j. \quad (21)$$

Now we can estimate  $u(z)$  for  $z \in B_j$ . By (18)–(21),

$$\begin{aligned} u(z) &\leq \int_{\partial\Omega^{(j)}} u^+(\zeta) \omega(z, d\zeta, \Omega^{(j)}) \\ &\leq C(B_j) \sum_{k=2}^3 \int_{a_j}^{b_j} u^+(t_k^{(j)} + r_k^{(j)}(y) + iy) d\nu(y) \\ &\quad + C(B_j) \sum_{m=1}^2 \int_{-1}^1 u^+(x + iy_m) dx \\ &\leq 2C(B_j)(M_1(\tau) + M_2(\lambda)), \end{aligned}$$

which completes the proof. □

## 5 Proof of Theorem 4

By Theorem 1 and Proposition 1, it suffices to prove

**Proposition 4** *If a function  $u$  satisfies the conditions of Theorem 4, then there exists a function  $f \in \mathcal{L}^-[-\pi, \pi]$  and a constant  $c_1 > 0$ , the both independent of  $u$ , such that*

$$\int_{-\pi}^{\pi} u^+(re^{i\theta})f(\theta) d\theta \leq c_1 V(r) \quad \forall r > r_0. \quad (22)$$

*Proof.* What we will do is a refinement of the arguments from the proof of the original Matsaev's theorem (see [15], [12]). Let

$$D_{r,R,a} = \{z \in \mathbb{C} : r < |z| < R, |\arg z - \pi/2| < \pi(1/2 - a)\}, \quad 0 < a < 1/4,$$

$b = (1 - 2a)^{-1}$ ,  $S(\theta, a) = \sin b(\theta - a\pi)$ . Carleman's formula for the function  $u$  harmonic in  $D_{r,R,a}$  has the form

$$\begin{aligned} & 2bR^{-b} \int_{\pi a}^{\pi-\pi a} u(Re^{i\theta})S(\theta, a) d\theta - b(r^{-b} + r^b R^{-2b}) \int_{\pi a}^{\pi-\pi a} u(re^{i\theta})S(\theta, a) d\theta \\ & - (r^{-b+1} - r^{b+1} R^{-2b}) \int_{-\pi a}^{\pi a} u'_r(re^{i\theta})S(\theta, a) d\theta \\ & + b \int_r^R [u(xe^{i\pi a}) + u(xe^{i\pi(1-a)})] (x^{-b-1} - x^{b-1} R^{-2b}) dx = 0. \end{aligned}$$

It implies the inequality

$$\begin{aligned} & \int_{\pi a}^{\pi-\pi a} u^+(Re^{i\theta})S(\theta, a) d\theta \leq c(r, u)R^b + \int_{\pi a}^{\pi-\pi a} u^-(Re^{i\theta})S(\theta, a) d\theta \\ & + R^b \int_r^R [u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})] (x^{-b-1} - x^{b-1} R^{-2b}) dx. \end{aligned} \quad (23)$$

Fix some  $\tau \in (0, 1/4)$  such that

$$\beta := (1 - 2\tau)^{-1} < 1 + \delta \quad (24)$$

with  $\delta$  as in the statement of Theorem 4. Inequality (23) gives us the relation

$$\begin{aligned} I_0 &:= \int_0^\tau \Phi(\sin \pi a) \int_{\pi a}^{\pi-\pi a} u^+(Re^{i\theta})S(\theta, a) d\theta da \\ &\leq c(r, u) \int_0^\tau R^b \Phi(\sin \pi a) da + \int_0^\tau \Phi(\sin \pi a) \int_{\pi a}^{\pi-\pi a} u^-(Re^{i\theta})S(\theta, a) d\theta da \\ &+ \int_0^\tau \Phi(\sin \pi a) \int_r^R [u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})] R^b x^{-b-1} dx da \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (25)$$

We can represent  $I_0$  as

$$I_0 = \int_0^\pi u^+(Re^{i\theta})\Psi(\theta) d\theta$$

with

$$\Psi(\theta) = \int_0^{\lambda(\theta)} S(\theta, a)\Phi(\sin \pi a) da \quad (26)$$

and

$$\lambda(\theta) = \min\{\theta/\pi, 1 - \theta/\pi, \tau\}. \quad (27)$$

Note that  $S(\theta, a) \geq 0$  when  $a \leq \lambda(\theta)$ , and  $S'_a(\theta, a) \leq 0$  for all  $a < 1/4$ . Since  $\Phi(t)$  is nondecreasing, this implies the bound

$$\Psi(\theta) \geq \int_{\lambda(\theta)/2}^{\lambda(\theta)} S(\theta, a)\Phi(\sin \pi a) da \geq f(\theta) = \lambda^2(\theta) \Phi\left(\sin \frac{\pi\lambda(\theta)}{2}\right)$$

and thus,

$$I_0 \geq \int_0^\pi u^+(Re^{i\theta})f(\theta) d\theta \quad (28)$$

with  $f \in \mathcal{L}^-[0, \pi]$ .

Let us now estimate the right hand side of (25). We have

$$I_1 \leq c(r, u)R^\beta \int_0^\tau \Phi(\sin \pi a) da \leq c_1(r, \tau, u)R^\beta; \quad (29)$$

$$I_2 = \int_0^\pi u^-(Re^{i\theta})\Psi(\theta) d\theta \leq \int_0^\pi u^-(Re^{i\theta})\Phi(\sin \theta) d\theta; \quad (30)$$

$$\begin{aligned} I_3 &\leq \int_0^\tau \int_r^R \Phi(\sin \pi a) [u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})] \left(\frac{R}{x}\right)^\beta x^{-1} dx da \\ &= R^\beta \int_r^R x^{-\beta-1} \left[ \int_0^{\pi\tau} + \int_{\pi(1-\tau)}^\pi \right] u^-(xe^{i\theta})\Phi(\sin \theta) d\theta dx \\ &\leq R^\beta \int_r^R x^{-\beta-1} \int_0^\pi u^-(xe^{i\theta})\Phi(\sin \theta) d\theta dx. \end{aligned} \quad (31)$$

We insert (28)–(31) into (25):

$$\begin{aligned} \int_0^\pi u^+(Re^{i\theta})f(\theta) d\theta &\leq c_1(r, \tau, u)R^\beta + \int_0^\pi u^-(Re^{i\theta})\Phi(\sin \theta) d\theta \\ &\quad + R^\beta \int_r^R x^{-\beta-1} \int_0^\pi u^-(xe^{i\theta})\Phi(\sin \theta) d\theta dx \\ &= J_1(R) + J_2(R) + J_3(R). \end{aligned} \quad (32)$$

By the choice of  $\beta$  (24),  $J_1(R) = o(V(R))$  as  $R \rightarrow \infty$ . Condition (7) implies  $J_2(R) \leq V(R)$ ,  $R > r_0$ . As to the term  $J_3$ , take any  $\epsilon \in (0, 1 + \delta - \beta)$ , then

$$\begin{aligned} J_3(R) &\leq R^\beta \int_r^R x^{-\beta-1} V(x) dx = R^\beta \int_r^R x^{-\beta-\epsilon} V(x) x^{\epsilon-1} dx \\ &\leq R^\beta R^{-\beta-\epsilon} V(R) \int_r^R x^{\epsilon-1} dx \leq \epsilon^{-1} V(R). \end{aligned}$$

These bounds give us

$$\int_0^\pi u^+(Re^{i\theta}) f(\theta) d\theta \leq c_2 V(R) \quad \forall R > r_1(u).$$

Absolutely the same way, we get a similar inequality in the lower half-plane and, as a result, relation (22).  $\square$

*Remark.* We do not know if condition (7) can be replaced by a more general one in terms of the class  $\mathcal{A}$ .

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Tek/Nat, University of Stavanger, 4036 Stavanger, Norway

E-MAIL: alexander.rashkovskii@uis.no